Polynomial Operators on Classes of Regular Languages

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CAI 2009

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The polynomial operator assigns to each class of languages $\mathscr{V}$ the class of all (positive) boolean combinations the languages of the form

$$L_0 a_1 L_1 a_2 \dots a_\ell L_\ell \ , \qquad (*)$$

where A is an alphabet, $a_1, \dots, a_\ell \in A$, $L_0, \dots, L_\ell \in \mathscr{V}(A)$ (i.e. they are over A).

The resulting classes are denoted by $\mathrm{PPol}\mathscr{V}$ and $\mathrm{BPol}\mathscr{V}$, respectively.

In the restricted case we fix a natural number k and we allow only $\ell \leq k$ in $(*)$. We get the classes $\mathrm{PPol}_k\mathscr{V}$ and $\mathrm{BPol}_k\mathscr{V}$, respectively.

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Basic examples

1. Let $\mathscr{T}(A) = \{\emptyset, A^*\}$ for each finite set A. Then PPol$\mathscr{T}$ is level 1/2 of the Straubing-Thérien hierarchy and BPol$\mathscr{T} = \mathscr{V}_1$ is level 1, i.e. the piecewise testable languages.

Result (Simon - 1972): Decidability of the membership problem for the class $\mathscr{V}_1$.

Open problem: Decidability of the membership problem for the class BPol$\mathscr{V}_1 = \mathscr{V}_2$.

2. Let $\mathscr{S}^+(A)$ be the set of all finite unions of the languages of the form B^*, where $B \subseteq A$, for each finite set A.

Result (Pin, Straubing): BPol$\mathscr{S}^+ = \mathscr{V}_2$.

Open problem – reformulation:

Is it decidable whether a given regular language $L \subseteq A^*$ can be expressed as a boolean combination languages of the form $B_0^* a_1 B_1^* a_2 \dots a_\ell B_\ell^*$, where $a_1, \dots, a_\ell \in A$, $B_0, \dots, B_\ell \subseteq A$.

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Other examples

3. Let $\mathscr{S}(A)$ be the set of all finite unions of the languages of the form $\overline{B}$, where $B \subseteq A$, for each finite set A. Here $\overline{B}$ is the set of all words over A containing exactly the letters from B.

4. Let m be a fixed natural number. Let $\mathscr{A}_m(A)$ be the set of all boolean combinations of the languages of the form $L(a,r) = \{ u \in A^* \mid |u|_a \equiv r \pmod{m} \}$, where $a \in A$ and $0 \leq r < m$, for each finite set A.

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Varieties of languages

A boolean variety of languages $\mathscr{V}$ associates to every finite alphabet A a class $\mathscr{V}(A)$ of regular languages over A in such a way that

- $\mathscr{V}(A)$ is closed under finite unions, finite intersections and complements (in particular, $\emptyset, A^* \in \mathscr{V}(A)$),
- $\mathscr{V}(A)$ is closed under derivatives, i.e.
  $L \in \mathscr{V}(A),\ u, v \in A^*$ implies
  $u^{-1}Lv^{-1} = \{ w \in A^* \mid uwv \in L \} \in \mathscr{V}(A)$,
- $\mathscr{V}$ is closed under inverse morphisms, i.e.
  $f : B^* \to A^*,\ L \in \mathscr{V}(A)$ implies
  $f^{-1}(L) = \{ v \in B^* \mid f(v) \in L \} \in \mathscr{V}(B)$.

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Pseudovarieties and varieties of monoids

A pseudovariety of finite (ordered) monoids is a class of finite monoids closed under submonoids, morphic images and products of finite families. Similarly for ordered monoids. When defining a variety of (ordered) monoids we use arbitrary products.

The pseudovarieties of ordered monoids can be characterized by pseudoidentities. The pseudovarieties we consider here are equational – they are given by identities, or equivalently, they are of the form Fin $\mathbf{V}$ where $\mathbf{V}$ is a variety of (ordered) monoids.

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Syntactic structures

For a regular language $L \subseteq A^*$, we define the relations $\sim_L$ and $\preceq_L$ on A^* as follows: for $u, v \in A^*$ we have

$$u \sim_L v \text{ if and only if } (\forall\, p, q \in A^*)\ (puq \in L \Longleftrightarrow pvq \in L),$$

$$u \preceq_L v \text{ if and only if } (\forall\, p, q \in A^*)\ (pvq \in L \Longrightarrow puq \in L).$$

The relation $\sim_L$ is the syntactic congruence of L on A^*. It is of finite index (i.e. there are finitely many classes), the quotient structure $\mathrm{M}(L) = A^*/\sim_L$ is called the syntactic monoid of L.

The relation $\preceq_L$ is the syntactic quasiorder of L and we have $\preceq_L \cap \succeq_L = \sim_L$. Hence $\preceq_L$ induces an order on $\mathrm{M}(L) = A^*/\sim_L$, namely: $u\sim_L \leq v\sim_L$ if and only if $u \preceq_L v$. We speak about the syntactic ordered monoid of L; we denote the structure by $\mathrm{O}(L)$.

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Result (Eilenberg, Pin)

Boolean varieties (positive varieties) of languages correspond to pseudovarieties of finite monoids (ordered monoids) . The correspondence, written $\mathscr{V} \longleftrightarrow \mathbf{V}$ *(*$\mathscr{P} \longleftrightarrow \mathbf{P}$*), is given by the following relationship: for* $L \subseteq A^*$ *we have*

$$L \in \mathscr{V}(A) \text{ if and only if } \mathrm{M}(L) \in \mathbf{V}$$

$$(\ L \in \mathscr{P}(A) \text{ if and only if } \mathrm{O}(L) \in \mathbf{P}\ ).$$

Examples

Pseudovarieties of (ordered) monoids corresponding to the classes $\mathscr{T}, \mathscr{S}^+, \mathscr{S}, \mathscr{A}_m$ consist exactly of all finite members of the following varieties:

$$\mathbf{T} = \mathrm{Mod}(\, x = y \,),\ \mathbf{S}^+ = \mathrm{Mod}(\, x^2 = x,\, xy = yx, 1 \leq x \,),$$

$$\mathbf{S} = \mathrm{Mod}(\, x^2 = x,\, xy = yx \,),\ \mathbf{A}_m = \mathrm{Mod}(\, xy = yx,\, x^m = 1 \,)\ .$$

The names for the (ordered) monoids of the pseudovarieties $\mathbf{T}, \mathbf{S}^+, \mathbf{S}, \mathbf{A}_m$ are trivial monoids (semilattices with the smallest element 1, semilattices and abelian groups of index m, respectively)

Finite characteristics

Let $X = \{x_1, x_2, \dots\}$. A relation γ on X^* is a finite characteristic if it satisfies the following conditions:
(i) γ is a quasiorder on X^*;
(ii) γ is compatible with the multiplication, i.e. for each $u, v, w \in X^*$ we have

$$u \,\gamma\, v \quad \text{implies} \quad uw \,\gamma\, vw, \quad wu \,\gamma\, wv \;;$$

(iii) γ is fully invariant, i.e. for each morphism $\varphi : X^* \to X^*$ and each $u, v \in X^*$ we have

$$u \,\gamma\, v \quad \text{implies} \quad \varphi(u) \,\gamma\, \varphi(v) \;;$$

(iv) for each finite subset Y of the set X, the set Y^* intersects only finitely many classes of $X^*/\gamma \cap \gamma^{-1}$.

A bit of Universal algebra

A relation γ on X^* satisfying conditions (i) – (iii) is called a fully invariant compatible quasiorder. It determines a variety $\mathbf{V}_\gamma = \mathrm{Mod}\ \gamma$ of ordered monoids. The operators Id and Mod are mutually inverse bijections between varieties of ordered monoids and fully invariant compatible quasiorders on X^*. Condition (iv) says that the finitely generated free ordered monoids in $\mathbf{V}_\gamma$ are finite, i.e. $\mathbf{V}_\gamma$ is locally finite.

The pseudovariety $\mathrm{Fin}\,\mathbf{V}_\gamma$ of all finite members from $\mathbf{V}_\gamma$ corresponds to the positive variety $\mathscr{V}_\gamma$ of languages by

$$L \in \mathscr{V}_\gamma(A) \text{ if and only if } O(L) \in \mathrm{Fin}\,\mathbf{V}_\gamma, \text{ for all finite } A\,.$$

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Finite characteristic of classes of languages

We say that γ is a finite characteristic of a class of languages $\mathscr{V}$ if γ is a finite characteristic and for each finite alphabet A:

$$L \in \mathscr{V}(A) \quad \text{if and only if} \quad \gamma_A \subseteq \sim_L \ .$$

(Here A can be identified with a subset of X and γ_A is the restriction of γ to A^*.)

Lemma. Let $\mathscr{V}$ be a class of languages and γ be a finite characteristic of $\mathscr{V}$. Then:
(i) $\mathscr{V}$ is equal to the positive variety of languages $\mathscr{V}_\gamma$;
(ii) if $\mathbf{V}$ is the pseudovariety of finite ordered monoids corresponding to $\mathscr{V}$ then $\gamma = \mathrm{Id}\langle \mathbf{V} \rangle$.

The classes of languages in our basic examples have the following finite characteristics:

1. $\mathrm{Id}\mathbf{T} = X^* \times X^*$.
2. $\mathrm{Id}\mathbf{S}^+ = \{(u,v) \in X^* \times X^* \mid \mathrm{c}(u) \subseteq \mathrm{c}(v)\}$.
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Classes of languages having a finite characteristic

Proposition

Let $\mathscr{V}$ be a positive variety of languages and let **V** *be the corresponding pseudovariety of ordered monoids. Then the following conditions are equivalent.*
(i) For each finite alphabet A, the set $\mathscr{V}(A)$ is finite.
(ii) The pseudovariety of ordered monoids **V** *is locally finite, i.e. each finitely generated submonoid of an arbitrary product of ordered monoids from* **V** *is finite.*
(iii) There exists a finite characteristic of $\mathscr{V}$.

Factorizations

Let k be a fixed natural number and γ be a finite characteristic. For a word $u \in X^*$, we say that

$$f = (u_0, a_1, \ldots, a_\ell, u_\ell)$$

is a factorization of u of length ℓ if $u_0, u_1, \ldots, u_\ell \in X^*$, $a_1, a_2, \ldots, a_\ell \in X$ and $u_0 a_1 u_1 \ldots a_\ell u_\ell = u$.

The set of all factorizations of lengths at most k of the word u is denoted by $\mathrm{Fact}_k(u)$.

Main construction

For a factorization $f = (u_0, a_1, u_1 \ldots, a_\ell, u_\ell)$ of a word $u \in X^*$ and a factorization $g = (v_0, b_1, v_1, \ldots, b_m, v_m)$ of a word $v \in X^*$, we write $f \leq_\gamma g$ if

- $\ell = m$,
- $a_i = b_i$ for every $i \in \{1, \ldots, \ell\}$,
- $u_i \ \gamma \ v_i$ for every $i \in \{0, 1, \ldots, \ell\}$.

We define the relation $\mathrm{p}_k(\gamma)$ on the set X^* as follows: for $u, v \in X^*$, we have $(u, v) \in \mathrm{p}_k(\gamma)$ iff

$$(\forall\, g \in \mathrm{Fact}_k(v))\,(\exists\, f \in \mathrm{Fact}_k(u))\; f \leq_\gamma g\ .$$

Main theorem

Theorem

Let $\mathscr{V}$ be a locally finite positive variety of languages and γ be a finite characteristic of $\mathscr{V}$. Then $\mathrm{PPol}_k\mathscr{V}$ is a locally finite positive variety of languages with the finite characteristic $\mathrm{p}_k(\gamma)$ and $\mathrm{BPol}_k\mathscr{V}$ is a locally finite boolean variety of languages with the finite characteristic $\mathrm{p}_k(\gamma) \cap \mathrm{p}_k(\gamma)^{-1}$.

Basic problems

- Relationships among classes (e.g. $\mathrm{PPol}_k\mathscr{S}^+$, $\mathrm{PPol}_k\mathscr{S}$)
- Bases of identities
- Generating by a single structure
- Impact to level 2

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Generating by a single ordered monoid

Let γ be a finite characteristic. We say that γ is finitely determined if there is a finite alphabet A such that for every finite alphabet B and all $u, v \in B^*$ we have:

$$((\forall\, \varphi : B \to A) \; \varphi(u) \; \gamma_A \; \varphi(v)) \quad \text{implies} \quad u \; \gamma_B \; v \; .$$

Proposition

The following properties for a positive variety $\mathscr{V}$ and the corresponding pseudovariety **V** *are equivalent.*

- *$\mathscr{V}$ is generated by a finite number of languages.*
- **V** *is generated by a single ordered monoid.*
- *There exists a finite characteristic of $\mathscr{V}$ which is finitely determined.*

Applications

Proposition

The positive variety $\mathrm{PPol}_k\mathscr{S}^+$ is generated by a finite number of languages.

For $|A| = 2^{2k+1}$.

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