Automatic structures: simple, difficult, rather unsolvable, and very unsolvable problems

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Foundations

Automatic structures

Definition (Khoussainov & Nerode '95)

A relational structure $(V, (R_i)_{1 \leq i \leq n})$ is

1. regular, if $V \subseteq \Gamma^*$ and $R_i \subseteq V^k \subseteq (\Gamma^*)^k$ can be accepted by synchronous k-tape automata M and M_i, resp.
   regular structure $\mathcal{A}(P)$ given by presentation
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Examples of automatic structures

- all finite structures
- complete binary tree
- Presburger arithmetic $(\mathbb{N}, +)$ (Skolem arithmetic $(\mathbb{N}, \cdot)$ is not)
- $(\mathbb{Q}, \leq)$ (K '03: even automatic-homogeneous)

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Examples of automatic structures

- rewrite graph $(\Sigma^*, \rightarrow)$ of semi-Thue system
- configuration graph of Turing machines
- configuration graph with reachability $(Q\Gamma^*, \rightarrow, \rightarrow^*)$ of pushdown automata

Examples

- Cayley-graphs of automatic monoids, in particular of
  - rational monoids (Sakarovitch '87)
  - virtually free f.g., virtually free Abelian f.g., and of hyperbolic groups (Epstein et al. '92)
  - singular Artin monoids of finite type (Corran, Hoffmann, K & Thomas '06)
  - graph products of such monoids (Fohry & K '05)

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  - graph products of such monoids (Fohry & K '05)
- ordinal α automatic iff $\alpha < \omega^\omega$ (Delhommé, Goranko & Knapik '03)
- $\mathcal{B}$ = Boolean algebra of (co-)finite subsets of $\mathbb{N}$
  infinite Boolean algebra automatic iff $\mathcal{B}^n$ for some $n \in \mathbb{N}$ (Khoussainov, Nies, Rubin, Stephan '04)
- field automatic iff finite (Khoussainov, Nies, Rubin, Stephan '04)
- f.g. group automatic iff virtually Abelian (Oliver & Thomas '05)

Foundations

Difficult problems

- First-order logic
- Second-order logic
- Algorithmic complexity

Simple problems

- Bounded degree
- Ramsey's theorem

Rather unsolvable problems

Very unsolvable problems

Summary

First-order logic

Theorem (Büchi '60, Hodgson '82)

There is an algorithm with:

Input: presentation P and first-order formula $\varphi(\overline{x})$

Output: synchronous multi-tape automaton M_φ with $L(M_\varphi) = \{\overline{a} \mid \mathcal{A}(P) \models \varphi(\overline{a})\}$.

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Proof

set of relations accepted by synchronous multi-tape automata is effectively closed under cylindrification, Boolean operations, and projections

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Corollary

The first-order theory of automatic structures is uniformly decidable.

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Remark

Holds likewise for extension FO^+ with quantifiers $\exists^\infty$ and $\exists^{(p,q)}$
(Blumensath '99; Khoussainov, Rubin & Stephan '04)
but not for second-order logic, fixpoint logic, ...

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The logic FSO

extend set of formulas of FO^+ by formation rule
R n-ary relation variable, $R(\overline{y})$ occurs only negatively in formula φ
$\Longrightarrow \exists R$ infinite : φ formula

Remark

then $\forall R, S(\varphi(R \cup S)) \to \varphi(R))$ tautology

Example

$\exists X$ infinite $\forall x, y : x, y \in X \to (x, y) \in E$
expresses "graph (V, E) has an infinite clique"

The logic FSO – continued

Theorem (K & Lohrey '08)

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Proof

use coding of certain infinite sets by ω-words and theory of ω-automatic structures

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Corollary

The FSO-theory of automatic structures is uniformly decidable.

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Complexity

- Σ_{N+1} is set of formulas

$$\exists \overline{x}_1 \forall \overline{x}_2 \ldots \exists/\forall \overline{x}_{N+1} : \psi(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{N+1})$$

  with ψ quantifier-free.
- decision procedure by Büchi and Hodgson requires space $\exp(N, |\varphi| + |P|)$ for $\varphi \in \Sigma_{N+1}$
  Question: complexity caused by structure or by formula?

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$\forall N \in \mathbb{N} \, \exists \varphi_N \in \Sigma_{N+1}$
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Conjecture (this post-proceedings?)

$\exists$ automatic structure $\mathcal{A}$

$$\forall N \in \mathbb{N} : \Sigma_{N+1}\text{-theory of } \mathcal{A} \text{ is } N\text{-EXPSPACE-hard}$$

Bounded degree

Definition

$\mathcal{A} = (V, (R_i)_{1 \leq i \leq n})$ relational structure
define $E \subseteq V \times V$ by
$(u, v) \in E \iff \exists \overline{a} \in \bigcup_{1 \leq i \leq n} R_i$: u and v occur in tuple $\overline{a}$
$G(\mathcal{A}) = (V, E)$ is Gaifman-graph of $\mathcal{A}$

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$\mathcal{A}$ has bounded degree, if $G(\mathcal{A})$ has bounded degree

Bounded degree – continued

Theorem (K & Lohrey '09)

1. There is a 2EXPSPACE-algorithm with:
   Input: first-order formula φ and
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2. There is an automatic structure of bounded degree whose first-order theory is 2EXPSPACE-hard.

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Theorem (Ramsey '30)

Any infinite graph contains

(a) an infinite clique or

(b) an infinite discrete induced subgraph.

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Questions

1. How difficult is it to determine whether (a) or (b) holds?
2. Are there necessarily "simple" infinite cliques or "simple" infinite discrete induced subgraphs?

Arithmetical and analytical hierarchy

universe $\mathcal{U}$:	all finitary objects (e.g. natural numbers, words, automata, finite sets . . .)
relations:	all decidable relations on $\mathcal{U}$

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Σ^1_1: all formulas $\exists X_1, \ldots, X_m : \varphi$ with φ first-order, X_i relation variable of arity n_i

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$L \subseteq \mathcal{U}$ belongs to Σ^j_i : membership in L expressed by some Σ^j_i

Infinite cliques

$L =$ all pairs (M_1, M_2) of TM that describe graph with infinite clique

$(M_1, M_2) \in L$ iff

$$\begin{aligned} \mathcal{U} \models \exists X \subseteq A^* \exists Y \subset X \exists f \subseteq A^* \times A^* :& \\ & \forall u \in X : u \in L(M_1) \wedge \\ & \forall u, v \in X : (u, v) \in L(M_2) \wedge \\ & f \text{ is bijection from } X \text{ onto } Y \end{aligned}$$

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$\Rightarrow$ existence of an infinite clique in a recursive graph is in Σ^1_1

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Existence of an infinite clique in a recursive graph is Σ^1_1-complete.
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Existence of an infinite clique in a recursive graph is Σ_1^1-complete. Hence there exists a recursive graph without recursive infinite clique and without recursive infinite discrete induced subgraph.

Theorem (Rubin '07)

Existence of an infinite clique in an automatic graph is decidable and a regular infinite clique (discrete induced subgraph, resp.) can be computed.

Proof

follows from results on FSO.

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$\Longrightarrow$ existence of an Eulerian path in a recursive graph is in Σ^1_1

Eulerian recursive graphs

Theorem (Erdős, Grünwald & Vazsonyi '38)

A countable graph $G = (V, E)$ is Eulerian iff

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2. G has a vertex of odd or infinite degree
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   formula is truly marvellous, but this margin is too narrow to contain it.

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Existence of an Eulerian path in a recursive graph is complete for $D^0_3 = \{K \setminus L \mid K, L \in \Sigma^0_3\}$

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Proof "in Π_2^0"

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Theorem (K & Lohrey '08)

Σ_1^1-hard even for planar automatic graphs of bounded degree

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  Question: Is there an even stronger logic?
- Complexity of $\mathrm{FO}^{(+)}$-theory elementary for bounded degree
  Question:
  - Does this hold for FSO?
  - Do weaker restrictions suffice?
- different complexities for automatic and for recursive structures:
  - decidable vs. Σ^1_1-complete (clique, infinite set cover, ...)
  - different levels of arithmetical hierarchy (Eulerian path, number of ends)
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- Question: What about (ω-)(tree-)automatic structures?