Iteration grove theories with applications

Dedicated to Prof. Werner Kuich

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Outline

1. Motivation
   - Continuous functions over a complete lattice
2. Identities of iteration
   - Main Results
   - Fixed point induction

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Continuous functions over a complete lattice

Consider the category $Cont_L$:

- objects: the natural numbers
- morphisms $n \to p$: the continuous functions $L^p \to L^n$
- †: taking morphisms $n \to n + p$ to morphisms $n \to p$
- $f^\dagger(y) := \bigwedge\{z : f(z, y) \leq z\}$
- $f^\dagger(y)$ is the least solution of $f(z, y) = z$
- *$Cont_L$ is an iteration theory!*

Continuous functions over a complete lattice

Dagger

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- $f^\dagger(y)$ is the smallest solution of $f(z,y) = z$

Define an another operation on $Cont_L$:

(Generalized) Star

- $^\otimes$: taking morphisms $n \longrightarrow n+p$ to morphisms $n \longrightarrow n+p$
- $f^\otimes(x,y) := \bigwedge\{z : x \leq f(z,y) \leq z\}$
- $f^\otimes(x,y)$ is the least solution of $f(z,y) = z$ over x

Continuous functions over a complete lattice

Dagger

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Dagger and star are related!

Recall that $f : n \longrightarrow n+p$ in $Cont_L$ is a continuous function $f : L^{n+p} \longrightarrow L^n$!

- for $f^{\tau} : L^{n+n+p} \longrightarrow L^n$ mapping (z, x, y) to $f(z, y) \vee x$:
- $f^{\otimes}(x, y) = (f^{\tau})^{\dagger}(x, y)$ and $f^{\dagger}(y) = f^{\otimes}(\bot, y)$.

(Grove) Theories

A *Theory* is:

- an $\mathbb{N}_0$ category,
- n is the n-fold coproduct of 1 with itself,
- so, there are tupling and separated sum operations, denoted: $\langle f, g \rangle$ and $(f \oplus g)$.

Grove theory: we have an additional sum operation: $+$

- associativity, commutativity,
- a neutral element: $f + 0_{n,p} = f = 0_{n,p} + f$, for all $n, p \geq 0$,
- distributivity on the right: $(f + g) \bullet h = (f \bullet h) + (g \bullet h)$,
- zero is left annihilator: $0_{n,p} \bullet h = 0_{n,q}$ for all $h : p \to q$.

Example: $Cont_L$ is a grove theory

(Generalized) Star

Up to this point we have:

- $Cont_L$ is an iteration grove theory
- star and dagger in $Cont_L$ are related

Main Question: What is the relationship between the equational properties of dagger and star in (iteration) grove theories?

Outline

Identities of iteration

$$f^{\dagger}(y) = f^{\otimes}(\perp, y) \text{ and } f^{\otimes}(x,y) = (f^{\tau})^{\dagger}(x,y) \text{ expressed as:}$$
$$f^{\dagger} = f^{\otimes} \bullet \langle 0_{n,p}, \mathbf{1}_p \rangle \text{ and } f^{\otimes} = (f^{\tau})^{\dagger}, \text{ where}$$
$$f^{\tau} = f \bullet (\mathbf{1}_n \oplus 0_n \oplus \mathbf{1}_p) + (0_n \oplus \mathbf{1}_n \oplus 0_p)$$

- fixed point: $f^{\dagger} = f \bullet \langle f^{\dagger}, \mathbf{1}_p \rangle$
- parameter: $(f \bullet (\mathbf{1}_n \oplus g))^{\dagger} = f^{\dagger} \bullet g$
- left zero: $(0_n \oplus f)^{\dagger} = f$
- right zero: $(f \oplus 0_q)^{\dagger} = f^{\dagger} \oplus 0_q$

- star fixed point: $f^{\otimes} = f \bullet \langle f^{\otimes}, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p)$
- star parameter: $(f \bullet (\mathbf{1}_n \oplus g))^{\otimes} = (f^{\otimes} \bullet (\mathbf{1}_n \oplus g))$
- star left zero: $(0_n \oplus f)^{\otimes} = (0_n \oplus f) + (\mathbf{1}_n \oplus 0_p)$
- star right zero: $(f \oplus 0_q)^{\otimes} = f^{\otimes} \oplus 0_q$

Identities of iteration

- double dagger:

$$f^{\dagger\dagger} = (f \bullet (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger$$

- double star:

$$(f^\otimes \bullet (\pi \oplus \mathbf{1}_p))^\otimes \bullet \langle 0_{n,n+p}, \mathbf{1}_{n+p} \rangle = (f \bullet (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\otimes$$

where $\pi = \langle 0_n \oplus \mathbf{1}_p, \mathbf{1}_n \oplus 0_p \rangle$

Identities of iteration

- composition:

$$(f \bullet \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\dagger = f \bullet \langle (g \bullet \langle f, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle$$

- star composition:

$$(f \bullet \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\otimes =$$
$$f^\tau \bullet \langle (g \bullet \langle f^\tau, 0_{m+n} \oplus \mathbf{1}_p \rangle^\otimes, 0_m \oplus \mathbf{1}_{n+p} \rangle \bullet \langle 0_{m,n+p}, \mathbf{1}_{n+p} \rangle$$

where $f^\tau = f \bullet (\mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p) + (0_m \oplus \mathbf{1}_n \oplus 0_p)$

Identities of iteration

- permutation:

$$(\pi \bullet f \bullet (\pi \oplus \mathbf{1}_p))^\dagger = \pi \bullet f^\dagger \bullet (\pi^{-1} \oplus \mathbf{1}_p)$$

- star permutation:

$$(\pi \bullet f \bullet (\pi^{-1} \oplus \mathbf{1}_p))^\otimes = \pi \bullet f^\otimes \bullet (\pi^{-1} \oplus \mathbf{1}_p)$$

here π is a tupling of coproduct injections which is an isomorphism

Identities of iteration

- pairing (or *Bekic'*)

$$\langle f, g\rangle^\dagger = \langle f \bullet \langle h^\dagger, \mathbf{1}_p\rangle, h^\dagger\rangle$$

  where $h = g \bullet \langle f^\dagger, \mathbf{1}_{m+p}\rangle$

- star pairing:

  $\langle f, g\rangle^\otimes =$
  $\langle f^\otimes \bullet \langle \mathbf{1}_n \oplus 0_{m+p}, k^\otimes \bullet (\pi^{-1} \oplus \mathbf{1}_p)\rangle, 0_{n+m} \oplus \mathbf{1}_p\rangle, k^\otimes \bullet (\pi^{-1} \oplus \mathbf{1}_p)\rangle$

  where $k = g \bullet \langle f^\otimes \bullet (\pi \oplus \mathbf{1}_p), \mathbf{1}_m \oplus 0_n \oplus \mathbf{1}_p\rangle$ and
  $\pi = \langle 0_m \oplus \mathbf{1}_n, \mathbf{1}_m \oplus 0_n\rangle$

Identities of iteration

- group identities:

$$g_S^\dagger = \tau_n \bullet (g \bullet (\tau_n \oplus \mathbf{1}_p))^\dagger$$

- star group identities:

$$g_S^\otimes \bullet (\tau_n \oplus \mathbf{1}_p) = \tau_n \bullet (g \bullet (\tau_n \oplus \mathbf{1}_p))^\otimes$$

the identities are associated to each finite group S, here τ_n is the unique base morphism $n \longrightarrow 1$ and $g_S = \langle g \bullet (\rho_1^S \oplus \mathbf{1}_p), \ldots, g \bullet (\rho_n^S \oplus \mathbf{1}_p)\rangle$ where $\rho_i^S = \langle (i \circ 1)_n, \ldots, (i \circ n)_n \rangle$

Iteration (Grove) Theories

- all dagger identities listed except the group identities:
  Conway grove theory
- Conway identities + dagger group identities:
  iteration grove theory

- all star identities listed except the group identities:
  Conway star theory
- Conway star identities + star group identities:
  iteration star theory

Completeness

Theorem (Ésik, 2000)

An equation between dagger terms holds in all iteration grove theories satisfying $(1_2 + 2_2)^\dagger = \mathbf{1}_1$ iff it holds in all theories $Cont_L$, for all complete lattices L.

So, we have the following **completeness** result:

Theorem

An equation between star terms holds in all iteration star theories satisfying $\mathbf{1}_1^\otimes = \mathbf{1}_1$ iff it holds in all theories $Cont_L$, for all complete lattices L.

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Fixed point induction

- Idempotent grove theory: $f + f = f$
- Define: $f \leq g \iff f + h = g$ for some $h \iff f + g = g$
- In idempotent grove theories $\leq$ is preserved by composition, sum, tupling, etc...
- The fixed point induction rule holds if
  $f \bullet \langle g, \mathbf{1}_p \rangle \leq g \Longrightarrow f^\dagger \leq g$,
  for $f : n \to n + p$, $g : n \to p$
- The star fixed point induction rule holds if
  $f \bullet \langle g, 0_n \oplus \mathbf{1}_p \rangle + h \leq g \Longrightarrow f^\otimes \bullet \langle h, 0_n \oplus \mathbf{1}_p \rangle \leq g$
  for $f, g, h : n \to n + p$

Fixed point induction

Proposition

- The fixed point induction rule: $f \bullet \langle g, \mathbf{1}_p \rangle \leq g \implies f^{\dagger} \leq g$ holds in all theories $Cont_L$.
- The star fixed point induction rule:
  $f \bullet \langle g, 0_n \oplus \mathbf{1}_p \rangle + h \leq g \implies f^{\otimes} \bullet \langle h, 0_n \oplus \mathbf{1}_p \rangle \leq g$ holds as well.

Fixed point induction

Theorem (Ésik, 2000)

An equation between dagger terms holds in all theories $Cont_L$ iff it holds in all idempotent grove theories which are dagger theories satisfying the fixed point identity, the parameter identity, and the fixed point induction rule.

Theorem

An equation between star terms holds in all theories $Cont_L$ iff it holds in all idempotent grove theories equipped with a star operation satisfying the star fixed point identity, the star parameter identity, and the star fixed point induction rule.

Examples

- Every matrix theory is a grove theory
- In matrix theories over ω-complete semirings we have:

$$[A,B]^{\otimes} = [A^{\star}, A^{\star}B] = [\sum A^k, \sum A^k B]$$

Consider the star fixed point:

$$[A,B]^{\otimes} = [A,B]\langle [A,B]^{\otimes}, 0_n \oplus \mathbf{1}_p\rangle + (\mathbf{1}_n \oplus 0_p)$$

is equivalent to

$$[A^{\star}, A^{\star}B] = [AA^{\star} + \mathbf{1}_n, AA^{\star}B + B]$$

is equivalent to

$$A^{\star} = AA^{\star} + \mathbf{1}_n$$

Examples

$Lang_\Sigma$

- the morphisms $n \longrightarrow p$ are n-tuples of tree languages $L \subseteq T_\Sigma(X_p)$
- addition is componentwise union
- composition is defined using OI substitution
- $L^\dagger$ is a least solution of $X = L \bullet \langle X, \mathbf{1}_p \rangle \Longrightarrow$ iteration grove theory
- $L^\otimes$ is a least solution of $Y = L \bullet \langle Y, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p) \Longrightarrow$ iteration star theory

Examples

$ST(A)$

- the morphisms $n \longrightarrow p$ are n-tuples of synchronization trees
- addition is merging of roots
- composition is substituting exit edges with trees
- $t^\dagger$ is the initial solution of $X = t \bullet \langle X, \mathbf{1}_p \rangle \Longrightarrow$ iteration grove theory
- $t^\otimes$ is the initial solution of $Y = t \bullet \langle Y, 0_n \oplus \mathbf{1}_p \rangle + (\mathbf{1}_n \oplus 0_p)$ $\Longrightarrow$ iteration star theory
- a bisimulation can be defined on trees, which is a congruence preserving dagger and star

Examples

$S_\Sigma\langle\langle X\rangle\rangle$

- S is a continuous semiring
- the morphisms $n \to p$ are n-tuples of tree series from $S_\Sigma\langle\langle X_p\rangle\rangle$
- addition is sum of tree series
- composition is obtained by OI substitution of tree series
- $t^\dagger$ is the least solution of $X = t \bullet \langle X, \mathbf{1}_p\rangle \implies$ iteration grove theory
- $t^\otimes$ is the least solution of $Y = t \bullet \langle Y, 0_n \oplus \mathbf{1}_p\rangle + (\mathbf{1}_n \oplus 0_p) \implies$ iteration star theory

Thank You

Thank You!

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