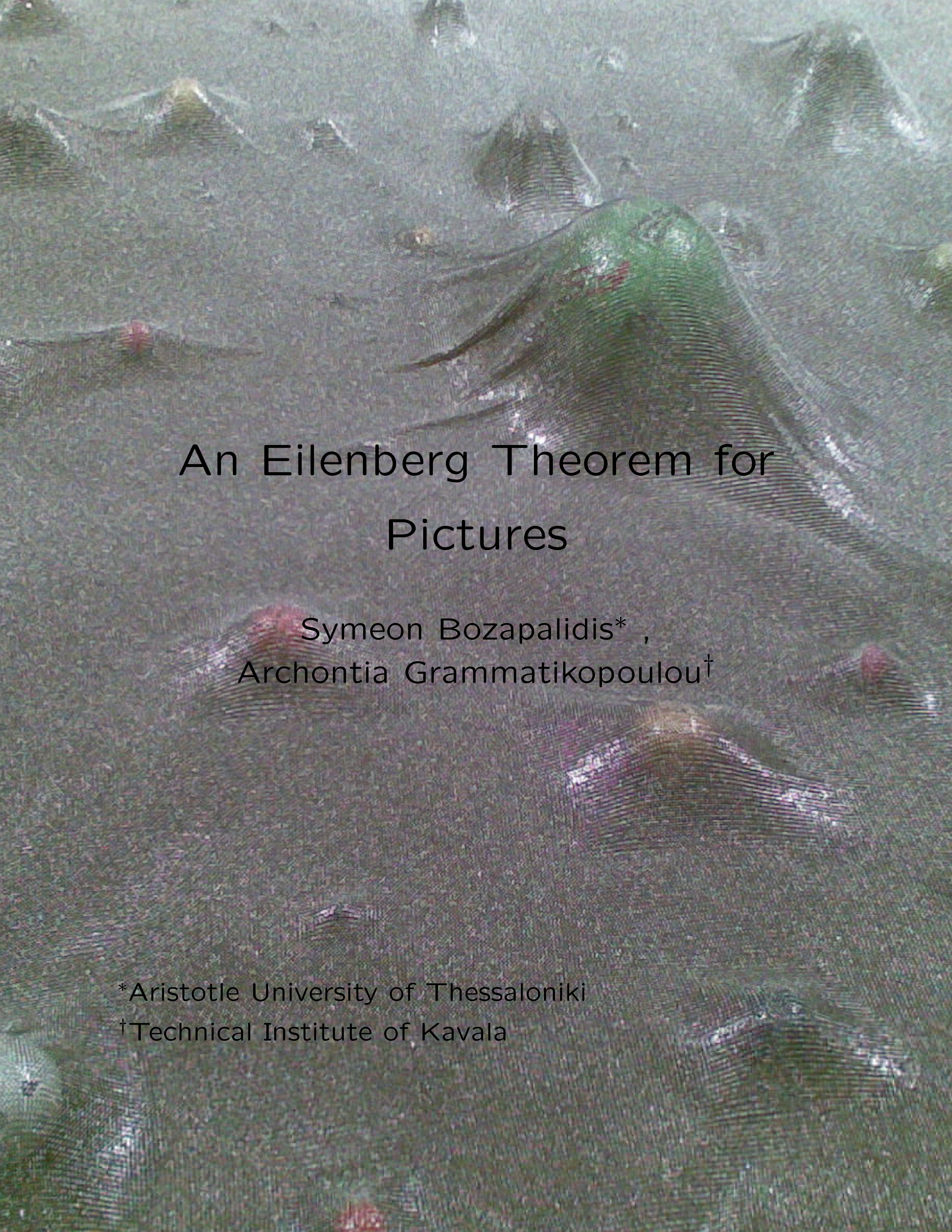


An Eilenberg Theorem for Pictures

Symeon Bozapalidis* ,
Archontia Grammatikopoulou†

*Aristotle University of Thessaloniki
†Technical Institute of Kavala

Dedicated to Werner Kuich for his retirement.

Recognizability through Monoid Action

A monoid is a set M equipped with an associative multiplication

$$M \times M \to M \ , \ (m_1, m_2) \mapsto m_1 m_2$$

which admits a unit element 1.

A set Q equipped with a function $M \times Q \to Q \ , \ (m, q) \mapsto m \cdot q$ such that

$$m_1(m_2 q) = (m_1 m_2)q \text{ and } 1 \cdot q = q$$

for all $q \in Q \ , \ m_1, m_2 \in M$ is called an M-*set*.

Given M-sets Q and Q', any function $h : Q \to Q'$ such that

$$h(m{\cdot}q) = m{\cdot}h(q) \qquad \text{for all } m \in M \ , \ q \in Q$$

is called an M-*function*.

The *left derivative* of $L \subseteq Q$ at $q \in Q$ is

$$q^{-1}L = \{m | m \in M \ , \ mq \in L\}.$$

The set of all left derivatives of L

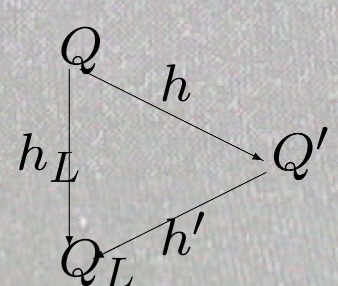
$$Q_L = \{q^{-1}L \mid q \in Q\}$$

with the (well defined) action

$$m(q^{-1}L) = (mq)^{-1}L \qquad (m \in M, q \in Q)$$

is structured into an M-set called the *syntactic M-set* of L.

Proposition 1. *Let L be a subset of an M-set Q. If $h : Q \to Q'$ is a surjective M-function such that $h^{-1}(h(L)) = L$, then there results a unique M-function $h' : Q' \to Q_L$ making commutative the triangle*



where $h_L : Q \to Q_L$ *is given by* $h_L(q) = q^{-1}L$.

The *right derivative* of $L \subseteq Q$ at $m \in M$ is

$$Lm^{-1} = \{q \mid q \in Q\ ,\ mq \in L\}.$$

Proposition 2. *If* L *has finitely many left derivatives, then it has finitely many right derivatives and vice versa.*

A subset L of an M-set Q is said to be *recognizable* if there is a finite M-set Q' and an M-function $h : Q \to Q'$ so that $L = h^{-1}(P)$, for some $P \subseteq Q'$.

Theorem 1. *Next conditions are equivalent for a subset* L *of the* M*-set* Q*:*

$i.$ L *is recognizable*

$ii.$ $card\{q^{-1}L \mid q \in Q\} < \infty$

$iii.\ card\{Lm^{-1} \mid m \in M\} < \infty$

Deformation Monoids

A *picture semigroup* is a family of sets $M = (M_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ equipped with two operations

$$ⓗ\colon M_{\alpha,\beta_1} \times M_{\alpha,\beta_2} \to M_{\alpha,\beta_1+\beta_2}$$

$$ⓥ\colon M_{\alpha_1,\beta} \times M_{\alpha_2,\beta} \to M_{\alpha_1+\alpha_2,\beta}$$

which are associative and compatible with each other, i.e.

$$(aⓗa')ⓥ(bⓗb') = (aⓥb)ⓗ(a'ⓥb')$$

In a picture semigroup $M = (M_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ we say that the family (e_α), $e_\alpha \in M_{\alpha,0}$ $(\alpha \in \mathbb{R}_+)$ is a *horizontal unit* whenever for all $a \in M_{\alpha,\beta}$ it holds that

$$e_\alpha \textcircled{h} a = a = a \textcircled{h} e_\alpha \quad \text{and} \quad e_\alpha \textcircled{v} e_\beta = e_{\alpha+\beta} \qquad .$$

The family of *vertical units* (f_β) are symmetrically defined.A picture semigroup with both horizontal and vertical units is called a *picture monoid*.

Assume that two picture monoids $M = (M_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ and $M' = (M'_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ are given. A *morphism of rank* (r, s), with $r, s \in \mathbf{R}_+ - \{0\}$, from M to M' is an $(\mathbf{R}_+ - \{0\})^2$-ranked family of functions

$$H = (H_{\alpha,\beta} : M_{\alpha,\beta} \to M'_{r\alpha,s\beta})_{\alpha,\beta\in\mathbf{R}_+}$$

preserving horizontal and vertical multiplications and units

$$\begin{aligned} H_{\alpha,\beta_1+\beta_2}(a \textcircled{h} a') &= H_{\alpha,\beta_1}(a) \textcircled{h} H_{\alpha,\beta_2}(a') \\ H_{\alpha_1+\alpha_2,\beta}(b \textcircled{v} b') &= H_{\alpha_1,\beta}(b) \textcircled{v} H_{\alpha_2,\beta}(b') \end{aligned}$$

$$H_{\alpha,0}(e_\alpha) = e'_\alpha \qquad H_{0,\beta}(f_\beta) = f'_\beta$$

where (e_α) , (f_β) (resp. (e'_α) , (f'_β)) are the horizontal and vertical units of M (resp. M').

The composition of two morphisms of ranks (r, s) and (r', s') respectively is a morphism of rank (rr', ss'). The morphisms of rank $(1, 1)$ are simply referred to as *morphisms of picture monoids*.

A 2-monoid is a structure $\mathbb{M}$ =(M,ⓗ,ⓥ,e,f) where ⓗ,ⓥ: $M^2 \to M$ are two associative operations admitting e,f respectively as unit elements. Moreover, we demand that ⓗ,ⓥ satisfy the coherence condition

$$(m_1 ⓗ m_2) ⓥ (m_1' ⓗ m_2') = (m_1 ⓥ m_1') ⓗ (m_2 ⓥ m_2')$$

for all $m_i, m_i' \in M$, $i = 1, 2$.

Furthermore, let $M = (M_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ be a picture monoid and $\sim = (\sim_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ be an equivalence relation on M compatible with horizontal and vertical multiplications

$$a \sim_{\alpha,\beta_1} a' \text{ and } b \sim_{\alpha,\beta_2} b' \quad \text{implies}$$
$$a ⓗ b \sim_{\alpha,\beta_1+\beta_2} a' ⓗ b'$$

$$a \sim_{\alpha_1,\beta} a' \text{ and } b \sim_{\alpha_2,\beta} b' \quad \text{implies}$$
$$a ⓥ b \sim_{\alpha_1+\alpha_2,\beta} a' ⓥ b'$$

Then we say that $\sim$ is a *congruence* on M.

The quotient ${}^{M}/_{\sim}$ can be organized into a picture monoid in the obvious way:

$$\overline{a} ⓗ \overline{b} = \overline{a ⓗ b} \quad , \quad \overline{a'} ⓥ \overline{b'} = \overline{a' ⓥ b'}$$

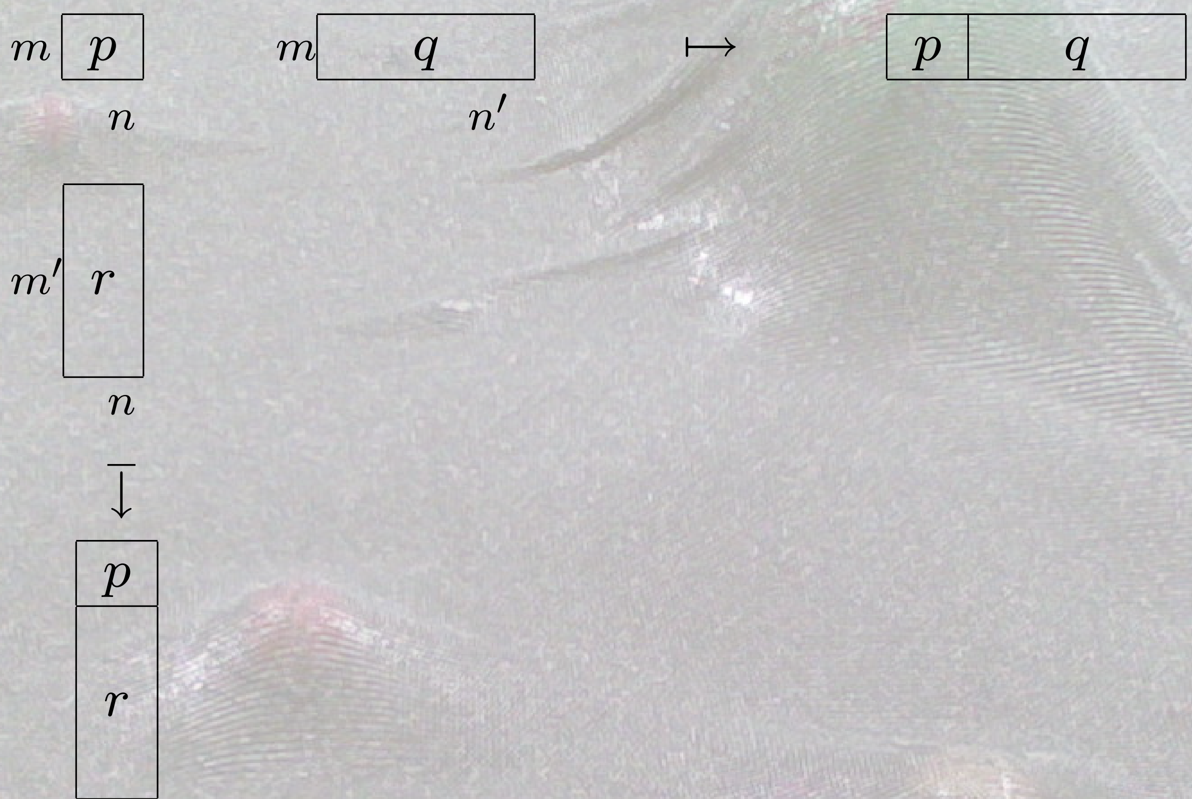
It is called the *quotient picture monoid* of M by $\sim$.

Let X be a finite (pixel) alphabet. A *picture of rank* (m,n) over X is just an $m \times n$ matrix with entries in X:

$$p = \begin{matrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{m1} & \cdots & x_{mn} \end{matrix} \qquad x_{ij} \in X.$$

We denote by $pict_{m,n}(X)$ the set of all such pictures.

Pictures can be composed in two ways: horizontally and vertically.



A *deformation monoid* (DM) is a pair $\mathcal{M} = (M, def_M)$ consisting of a picture monoid M and a family of morphisms of rank (r, s)

$$def_M^{(r,s)} : M \to M \qquad , \ r, s \in \mathbb{R}_+ - \{0\}$$

called the (r, s)-*deformation operator*, verifying the equalities:

$$def_M^{(r,s)} \circ def_M^{(r',s')} = def_M^{(rr',ss')} \quad , \ def_M^{(1,1)} = id_M.$$

A 2-monoid $\mathbb{M}$ can be viewed as a deformation monoid $N(\mathbb{M})$ by setting $N(\mathbb{M})_{\alpha,\beta} = M$ for all $\alpha, \beta \in \mathbb{R}_+$ whereas its deformation operators coincide with the identity function on M.

Given two deformation monoids $\mathcal{M} = (M, def_M)$ and $\mathcal{M}' = (M', def_{M'})$, any morphism of picture monoids $H : M \to M'$ commuting with deformation i.e.

$$def_{M'}^{(r,s)} \circ H_{\alpha,\beta} = H_{r\alpha,s\beta} \circ def_M^{(r,s)}$$

is termed a *morphism of deformation monoids* (DM morphism).

Denote by $P(X) = (P_{\alpha,\beta}(X))_{\alpha,\beta\in\mathbb{R}_+}$ the least $\mathbb{R}_+ \times \mathbb{R}_+$- indexed family of sets formally constructed by the following items:

i. $X \subseteq P_{1,1}(X)$

ii. if $p_1 \in P_{\alpha,\beta_1}(X)$ and $p_2 \in P_{\alpha,\beta_2}(X)$, then $p_1 p_2 \in P_{\alpha,\beta_1+\beta_2}(X)$

iii. if $p_1 \in P_{\alpha_1,\beta}(X)$ and $p_2 \in P_{\alpha_2,\beta}(X)$ then $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in P_{\alpha_1+\alpha_2,\beta}(X)$

iv. the horizontal and vertical *empty pictures* of rank $\alpha \in \mathbb{R}_+$

$$\varepsilon_\alpha \in P_{\alpha,0}(X) \text{ and } \zeta_\alpha \in P_{0,\alpha}(X)$$

play the role of units for the above two concatenations($\varepsilon_0 = \zeta_0$)

v. if $p \in P_{\alpha,\beta}(X)$, then $p^{(r,s)} \in P_{r\alpha,s\beta}(X)$.

The items $i. - iv.$ ensure that $P(X)$ is a picture monoid containing X whose operations are horizontal and vertical concatenations.

Consider the congruence $\sim$ on $P(X)$ generated by the relations

d_1. $p^{(1,1)} \sim p \quad , \quad (p^{(r,s)})^{(r',s')} \sim p^{(rr',ss')}$

d_2. $(p_1 p_2)^{(r,s)} \sim p_1^{(r,s)} p_2^{(r,s)}$

d_3. $\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^{(r,s)} \sim \begin{pmatrix} p_1^{(r,s)} \\ p_2^{(r,s)} \end{pmatrix}$

for all $p, p_1, p_2 \in P(X)$ of suitable rank. Then the quotient set $Pict(X) = {}^{P(X)}/_{\sim}$ is a deformation monoid: the deformation operation associated with $(r,s) \in (\mathbb{R}_+ - \{0\})^2$ is given by the mapping

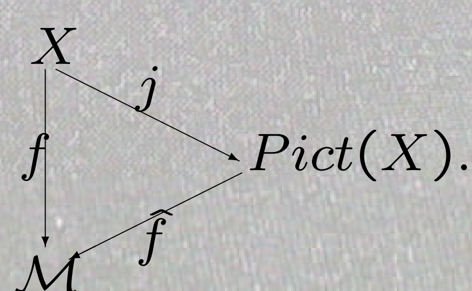
$$\overline{p} \ \mapsto \ \overline{p^{(r,s)}}$$

where $\overline{p}$ denotes the $\sim$-class of p.

Clearly any element p of $Pict_{\alpha,\beta}(X)$ can be represented by a picture of rank (α, β) constructed by the deformed pixels $x^{(r,s)}$ ($x \in X$, $r, s \in \mathbb{R}_+ - \{0\}$) whereas $p^{(r,s)}$ is the picture obtained by substituting any deformed pixel $x^{(\gamma,\delta)}$ in p by $x^{(r\gamma,s\delta)}$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}_+$.

A *picture language of rank* (α, β) over X is a subset of $Pict_{\alpha,\beta}(X)$ ($\alpha, \beta \in \mathbb{R}_+$). $Pict(X)$ is the free deformation monoid generated by X, as next theorem confirms.

Theorem 2. *The function* $j : X \to Pict_{1,1}(X)$, $j(x) = x$ *has the following universal property: for any deformation picture monoid* $\mathcal{M} = (M, def_M)$ *and any function* $f : X \to M_{1,1}$ *there is a unique morphism of deformation monoids* $\hat{f} : Pict(X) \to \mathcal{M}$ *rendering commutative the diagram*



The morphism $\widehat{f}$ is defined by the clauses:

- $\widehat{f}(x) = f(x) \quad , x \in X$

- $\widehat{f}(p_1 p_2) = \widehat{f}(p_1) ⓗ \widehat{f}(p_2)$

- $\widehat{f}\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right) = \widehat{f}(p_1) ⓥ \widehat{f}(p_2)$

- $\widehat{f}(p^{(r,s)}) = def_M^{(r,s)}(\widehat{f}(p))$

for all $p, p_1, p_2 \in P(X)$ *and* $(r, s) \in \mathbb{R}_+ - \{0\}$.

Frame Recognizability

Consider a pixel alphabet X and an auxiliary pixel $\xi \notin X$.

The *deformation equivalence* $\sim_{def}$ is defined on the set

$$\bigcup_{\alpha,\beta\in\mathbb{R}_+} Pict_{\alpha,\beta}(X)$$

as follows:

$$p \sim_{def} q \text{ iff } q = p^{(r,s)} \text{ for some } r,s \in \mathbb{R}_+ - \{0\}.$$

The set

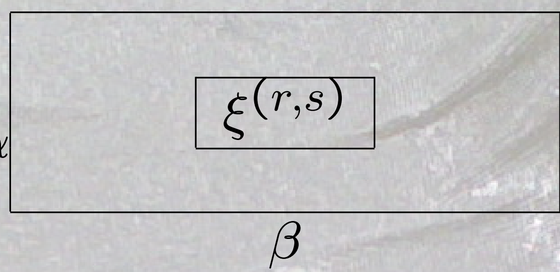
$$\mathfrak{Pict}(X) = (\bigcup_{\alpha,\beta\in\mathbb{R}_+} Pict_{\alpha,\beta}(X))/\sim_{def}$$

has as elements the $\sim_{def}$-classes of the pictures over X, i.e. the elements of $\mathfrak{Pict}(X)$ are of the form

$$\overline{p} = \{p^{(r,s)} \mid r,s \in \mathbb{R}_+ - \{0\}\}.$$

The set of *frames* with *exterior rank* (α,β) and *interior rank* (r,s) is the subset $Frame^{r,s}_{\alpha,\beta}(X)$ of $Pict_{\alpha,\beta}(X \cup \xi)$ consisting of all pictures with just one occurrence of a deformation of ξ, namely $\xi^{(r,s)}$

$f = \alpha$



$r \leqslant \alpha \ , \ s \leqslant \beta.$

Given frames

$f \in Frame^{r,s}_{\alpha,\beta}(X)$ and $f' \in Frame^{\gamma,\delta}_{r,s}(X)$

their composition $f \circ f'$ is the frame obtained by substituting f' at $\xi^{(r,s)}$ in f.

In general, if $f_i \in Frame^{r_i,s_i}_{\alpha_i,\beta_i}(X)$ we define the product

$$f_1 \cdot f_2 = f_1 \circ f_2^{(\frac{r_1}{\alpha_2},\frac{s_1}{\beta_2})}$$

For $f'_i \sim_{def} f_i$ ($i = 1, 2$), it holds that $f_1 \cdot f_2 \sim_{def} f'_1 \cdot f'_2$. It turns out that the quotient set

$$\mathfrak{Frame}(X) = \left(\bigcup_{\alpha,\beta \in \mathbb{R}_+ - \{0\}} Frame^{r,s}_{\alpha,\beta}(X) \right) / \sim_{def}$$

with multiplication $\overline{f_1} \cdot \overline{f_2} = \overline{f_1 \cdot f_2}$ becomes a monoid which canonically acts on $\mathfrak{Pict}(X)$: $\overline{f} \cdot \overline{p} = \overline{f \cdot p}$. In other words $\mathfrak{Pict}(X)$ is a $\mathfrak{Frame}(X)$-set and thus we can speak of recognizable subsets $\mathcal{L}$ of $\mathfrak{Pict}(X)$.

The left and right derivatives of $\mathcal{L} \subseteq \mathfrak{Pict}(X)$ are

$$\overline{p}^{-1}\mathcal{L} = \{\overline{f} \mid \overline{f} \in \mathfrak{Frame}(X)\ ,\ \overline{f} \cdot \overline{p} = \overline{f \cdot p} \in \mathcal{L}\}$$

$$\mathcal{L}\overline{f}^{-1} = \{\overline{p} \mid \overline{p} \in \mathfrak{Pict}(X)\ ,\ \overline{f} \cdot \overline{p} = \overline{f \cdot p} \in \mathcal{L}\}.$$

By applying Theorem 1 in the present setup we obtain

Proposition 3. *Next conditions are equivalent for a subset $\mathcal{L}$ of $\mathfrak{Pict}(X)$:*

i. there is a finite $\mathfrak{Frame}(X)$-set Q and a $\mathfrak{Frame}(X)$-function $h : \mathfrak{Pict}(X) \to Q$ so that $\mathcal{L} = h^{-1}(P)$, for some $P \subseteq Q$,

$ii.\ card\{\overline{p}^{-1}\mathcal{L} \mid \overline{p} \in \mathfrak{Pict}(X)\} < \infty$

$iii.\ card\{\mathcal{L}\overline{f}^{-1} \mid \overline{f} \in \mathfrak{Frame}(X)\} < \infty.$

We call $\mathcal{L} \subseteq \mathfrak{Pict}(X)$ *frame recognizable* whenever it satisfies one (and thus all) of the above conditions i-iii.

An immediate consequence of Proposition 3 concerns closure properties.

Proposition 4. *The frame recognizable subsets of $\mathfrak{Pict}(X)$ are closed under the boolean operations.*

A picture language $L \subseteq \mathfrak{Pict}(X)$ is said to be *deformation closed* whenever

$$p \in L \quad \text{and} \quad p' \sim_{def} p \qquad \text{implies} \qquad p' \in L.$$

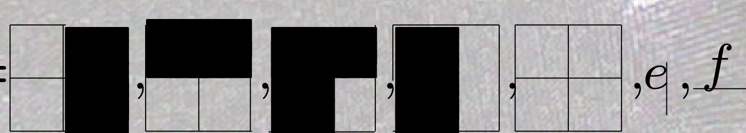
For a deformation closed picture language $L \subseteq Pict(X)$ its right and left derivatives are defined to be the corresponding derivatives of the associated set $\overline{L}$:

$$Lp^{-1} = \overline{L}\overline{p}^{-1}\ ,\ f^{-1}L = \overline{f}^{-1}L$$

for all $p \in Pict(X)$, $f \in Frame(X)$.

A deformation closed picture language $L \subseteq Pict(X)$ is called *frame recognizable* whenever $\overline{L} = \{\overline{p} \mid p \in L\} \subseteq \mathfrak{Pict}(X)$ is frame recognizable.

Example 1. *Let* $X = \{\square, \blacksquare\}$ *and consider the language* $L \subseteq Pict(X)$ *consisting of all pictures having black pixels along their north-western faces.* L *is obviously deformation-closed and has seven distinct left derivatives* $p^{-1}L$ *with:*

$p =$ 

where e, f *are the empty horizontal and vertical pixels respectively. Thus* L *is frame recognizable.*

Recognizability through Picture Monoids

In

O. Matz, *Regular Expressions and Context-free Grammars for Picture Languages*, Lecture Notes in Computer Science 1200, 1997, 283-294

Matz raised the question whether the word language recognizability through monoids can be transferred into the framework of pictures. In the present section we deal with this problem.

A deformation monoid $M = (M_{\alpha,\beta})_{\alpha,\beta\in\mathbb{R}_+}$ is said to be

- *locally finite* whenever the set $M_{\alpha,\beta}$ is finite for all indices $\alpha, \beta \in \mathbb{R}_+$

- *finite* whenever the set

$$\bigcup_{\alpha,\beta\in\mathbb{R}_+} M_{\alpha,\beta}$$

  is finite

- *reachable* if there is a finite pixel alphabet X and a deformation morphism $H : Pict(X) \to M$ which is locally surjective, i.e. all the functions $H_{\alpha,\beta} : Pict_{\alpha,\beta}(X) \to M_{\alpha,\beta}$ are surjective. This implies that any element $m \in M_{\alpha,\beta}$ $(\alpha,\beta \in \mathbb{R}_+)$ can be obtained from a list of elements $def_M^{(r_1,s_1)}(m_1)$, ... $,def_M^{(r_k,s_k)}(m_k)$ (with $m_1,...,m_k \in M_{1,1}$) by applying the operations of horizontal and vertical multiplication.

Since $\mathbb{R}_+ - \{0\}$ is a multiplicative group, the deformation operator $def_M^{(r,s)} : M_{\alpha,\beta} \to M_{r\alpha,s\beta}$

is bijective and its inverse is $def_M^{(\frac{1}{r},\frac{1}{s})} : M_{r\alpha,s\beta} \to M_{\alpha,\beta}$.

A picture language $L \subseteq Pict(X)$ is *recognizable* if there is a locally finite deformation monoid M and a deformation morphism $H : Pict(X) \to M$ so that $L = h^{-1}(R)$, with $R \subseteq M$ (i.e. $R_{\alpha,\beta} \subseteq M_{\alpha,\beta}$ for all $\alpha, \beta \in \mathbb{R}_+$).

Given a 2-monoid $\mathbb{M}$ =(M,ⓗ,ⓥ,e,f) a *morphism* from $Pict(X)$ to $\mathbb{M}$ is a family of functions

$$H_{\alpha,\beta} : Pict_{\alpha,\beta}(X) \to M \ \ , \ \alpha, \beta \in \mathbb{R}_+$$

which are compatible with horizontal, vertical concatenations and units

$$H_{\alpha,\beta_1+\beta_2}(p_1 p_2) = H_{\alpha,\beta_1}(p_1) ⓗ H_{\alpha,\beta_2}(p_2)$$
$$H_{\alpha_1+\alpha_2,\beta}\left(\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right) = H_{\alpha_1,\beta}(q_1) ⓥ H_{\alpha_2,\beta}(q_2)$$
$$H_{1,0}(\varepsilon) = e$$
$$H_{0,1}(\varphi) = f$$

and respect deformation

$$p \sim_{def} p' \quad \text{implies} \quad H_{\alpha,\beta}(p) = H_{\alpha',\beta'}(p').$$

Theorem 3. *A deformation closed language $L \subseteq Pict(X)$ is frame recognizable if and only if there exist a finite 2-monoid $\mathbb{M} = (M, ⓗ, ⓥ, e, f)$ and a morphism $H : Pict(X) \to \mathbb{M}$ so that*

$$L = H^{-1}(P) \quad , \text{ for some } P \subseteq M.$$

Syntactic 2-monoids

Since $\mathbb{R}_+ - \{0\}$ is a multiplicative group, all deformation operators of $Pict(\Sigma)$ are bijective and so if $H : Pict(\Sigma) \to M$ is a morphism of deformation monoids (M is a 2-monoid) then for all $r, s \in \mathbb{R}_+ - \{0\}$ the language $H^{-1}_{r,s}(Q)$, $Q \subseteq M$, is the (r, s)-deformation of the language $H^{-1}_{1,1}(Q)$. In other words $H^{-1}(Q)$ is completely determined by $H^{-1}_{1,1}(Q)$.

Next, given a deformation-closed picture language $L \subseteq Pict(\Sigma)$, the set of its right derivatives

$$M_L = \{Lp^{-1} \mid p \in Pict(\Sigma)\}$$

can be canonically converted into a 2-monoid by defining the horizontal and vertical multiplication via the formulas

$$(Lp_1^{-1})ⓗ(Lp_2^{-1}) = L(p_1p_2)^{-1}$$

$$(Lq_1^{-1})ⓥ(Lq_2^{-1}) = L\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^{-1}.$$

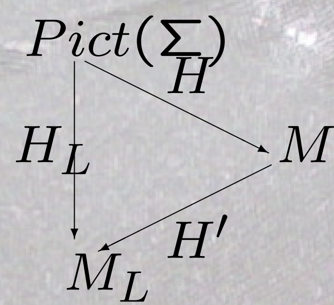
The canonical deformation morphism

$$H_L : Pict(\Sigma) \to M_L, H_L(p) = Lp^{-1}$$

is clearly surjective and verifies the equation $H^{-1}(H(L)) = L$.

Actually H_L is universal with the above property in the following sense.

Theorem 4. *Let $H : Pict(\Sigma) \to M$ be a deformation epimorphism (M a 2-monoid) such that $H^{-1}(H(L)) = L$. Then there exists a unique epimorpism of 2-monoids $H' : M \to M_L$ making commutative the triangle*



Given 2-monoids M, M' we write $M < M'$ whenever M is a surjective image of a 2-submonoid of M'.

With this notation, we have
Proposition 5. *Let $L_1, L_2, L \subseteq Pict(\Sigma)$ be deformation closed. Then*

$$M_{L_1 \cup L_2} < M_{L_1} \times M_{L_2}\,,\; M_{L_1 \cap L_2} < M_{L_1} \times M_{L_2}\,,$$

$$M_{L^c} = M_L\,,\; M_{\tau^{-1}L} < M_L$$

$\tau \in \mathfrak{Frame}(\Sigma)$, *where* L^c *is the set theoretic complement of* L.

Moreover, if $F : Pict(\Delta) \to Pict(\Sigma)$ *is a homomorphism of deformation monoids, then* $M_{F^{-1}(L)} < M_L$.

Remark. *The theory of syntactic 2-monoids could be linked with the general theory of syntactic algebras, and especially with the many-sorted version presented in*

S. Salehi, M. Steinby, Varieties of many-sorted recognizable sets, *PU.M.A. 18, 2007, 319343.*

The Variety Theorem

We adapt the arguments of Eilenberg

S. Eilenberg, *Automata, Languages and Machines*, vol. B, Academic Press, New York, 1977

A class of finite 2-monoids closed under isomorphism is called a *pseudovariety* of 2-monoids whenever next axioms are fulfilled:

$pv1$) If $M_1, ..., M_k \in \mathbb{V}$, then $M_1 \times ... \times M_k \in \mathbb{V}$.

$pv2$) If $g : M' \to M$ is a monomorphism of 2-monoids and $M \in \mathbb{V}$, then $M' \in \mathbb{V}$.

$pv3$) If $h : M \to M''$ is an epimorphism of 2-monoids and $M \in \mathbb{V}$, then $M'' \in \mathbb{V}$.

Clearly $pv1$)+$pv2$) can be replaced by the single axiom

pv) If $M' < M$ and $M \in \mathbb{V}$ then $M' \in \mathbb{V}$.

The intersection of any family of pseudovarieties of 2-monoids is again a pseudovariety of

2-monoids so we can speak of the pseudovariety generated by a class V of finite 2-monoids. It is denoted by $< V >$.Clearly

Proposition 6. *It holds*

$$M \in < V > \quad \textit{iff} \quad M < M_1 \times ... \times M_k \ \textit{ with } M_i \in V.$$

Proposition 7. *Each pseudovariety $\mathbb{V}$ is generated by the syntactic 2-monoid it contains.*

Now, assume that for every finite pixel alphabet Σ, a family $\mathcal{P}(\Sigma)$ of frame recognizable picture languages is given, so that

$lv1)$ $\mathcal{P}(\Sigma)$ is closed under boolean operations(union, intersection, complement)

$lv2)$ $\mathcal{P}(\Sigma)$ is closed under right derivatives

$$\tau \in \mathfrak{Frame}(\Sigma) \ , \ L \in \mathcal{P}(\Sigma) \ \text{ implies } \ \tau^{-1}L \in \mathcal{P}(\Sigma)$$

$lv3)$ for any deformation homomorphism $F : Pict(\Delta) \to Pict(\Sigma)$ we have

$$L \in \mathcal{P}(\Sigma) \text{ implies } F^{-1}(L) \in \mathcal{P}(\Delta).$$

Then we say that the family $(\mathcal{P}(\Sigma))_\Sigma$ is a *variety of frame recognizable languages*.

Proposition 8. *Let $\mathcal{P} = (\mathcal{P}(\Sigma))_\Sigma$ be a variety of frame recognizable languages. If $L \in \mathcal{P}(\Sigma)$ and $p \in Pict(\Sigma)$ then the equivalence class of p, $[p] = H_L^{-1}(H_L(p))$ is also in $\mathcal{P}(\Sigma)$.*

To each variety $\mathcal{P} = (\mathcal{P}(\Sigma))_\Sigma$ of frame recognizable languages we associate the pseudovariety of 2-monoids $\mathbb{V}_\mathcal{P}$ which is generated by the syntactic 2-monoid of the languages of $\mathcal{P}$.

In the opposite direction, to each pseudovariety $\mathbb{V}$ of finite 2-monoids we associate the class $\mathcal{P}_\mathbb{V} = (\mathcal{P}_\mathbb{V}(\Sigma))_\Sigma$ such that $L \in \mathcal{P}_\mathbb{V}(\Sigma)$ iff

$M_L \in \mathbb{V}$. By virtue of Proposition 5, $\mathcal{P}_{\mathbb{V}}$ is a variety of frame recognizable picture languages.

Theorem 5. *The assignments*

$$\mathcal{P} \mapsto \mathbb{V}_{\mathcal{P}} \text{ and } \mathbb{V} \mapsto \mathcal{P}_{\mathbb{V}}$$

are mutually inverse to each other.

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